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Note

Concertina-Like Movement in the Absence of a Chebyshev System

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INTRODUCTION

Meinardus [1, p. 29] defined functions $S(x)$ having certain oscillatory and best approximation properties on an interval $[a, b]$. The most notable example is the Chebyshev polynomial of the first kind, $T_n(x)$. In [2], Streit studied the dependence of $S(x)$ on the left endpoint, a, of the interval and discussed an application to the design of linear antenna arrays. The dependence on the endpoint was further investigated by Zielke [3] who obtained stronger results. We will summarize briefly some of the theory and then present an example to settle a certain question.

PROPERTIES OF $S_t(x)$

Let $[a, b]$ be a finite real interval, n a positive integer and $h_1 = 1, h_2,..., h_n$, f real continuous functions on $[a, b]$ such that $\{1, h_2,..., h_n\}$ is a Chebyshev system of degree n on $[a, b]$ (i.e., $\sum_{i=1}^{n} a_i h_i$ has at most $n-1$ zeros in $[a, b]$ unless $a_1 = 0,..., a_n = 0$). Assume also that $\{1, h_2,..., h_n, f\}$ is a Chebyshev system of degree $n+1$ on $[a, b]$. Let $a \leqslant t < b$ and let $p_t(x)$ denote the best

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uniform approximation to $f(x)$ on $[t, b]$ by a linear combination of 1, $h_2,..., h_n$. Then [1, p. 29], $f-p_t$ has exactly $n+1$ extremals of alternating sign and equal magnitude which include the endpoints a and b, and $f-p_t$ is a strictly monotone function of x between these extremals. Define

$$S_t(x) = \pm[f(x) - p_t(x)] / \max_{t \leqslant x \leqslant b} |f(x) - p_t(x)|$$

where the sign is chosen so that $S_t(b) = +1$.

If $\{1, h_2,..., h_n, f\}$ is $\{1, x,..., x^n\}$ and $[a, b] = [-1, 1]$, then

$$S_t(x) = T_n\left(\frac{2x}{1-t} - \frac{1+t}{1-t}\right).$$

Motivated by results obtained from the application of the shifted Chebyshev polynomials to linear antenna arrays, Streit [2] studied for the general case the movement of the zeros and extremals of S_t as a function of t. In [4] Zielke showed the entire graph of S_t moves to the right as t increases (concertina-like movement) except possibly the extremal points. They, too, must move to the right if the derivatives $\{h_2',..., h_n', f'\}$ form a Chebyshev system of degree n on (a, b). Of course, the right-hand endpoint of the graph stays fixed at $(b, S_t(b)) = (b, 1)$. We summarize the known properties of S_t: For each t such that $a \leqslant t < b$,

(a) S_t is a linear combination of 1, $h_2,..., h_n, f$.

(b) $\max_{t \leqslant x \leqslant b} |S_t(x)| = 1$.

(c) The best uniform approximation to S_t on $[t, b]$ by a linear combination of $\{1, h_2,..., h_n\}$ is 0.

(d) $S_t(x)$ has $n+1$ extremals of alternating sign and equal magnitude, which include the endpoints t and b, and $S_t(x)$ is a strictly monotone function of x between the extremals.

(e) $S_t(b) = 1$.

(f) S_t satisfying (a)–(e) is unique.

(g) The graph of S_t moves to the right as t increases (except for the fixed right-hand endpoint); i.e., $a \leqslant t_1 < t_2 < b$, α in $[-1, 1]$, and $1 \leqslant k \leqslant n$ implies that the smallest z such that $S_{t_1}(x) = \alpha$ for k distinct points in $[t_1, z]$ is strictly less than the smallest z such that $S_{t_2}(x) = \alpha$ for k distinct points in $[t_2, z]$.

The Example

Proof of the existence of S_t with the nice properties (a)–(g) relies heavily on the fact that $\{1, h_2,..., h_n, f\}$ is a Chebyshev system. We were curious as to whether a system could give rise to an S_t satisfying (a)–(g) without being a Chebyshev system. Clearly this is impossible for $\{1, f\}$ since $c_1 f - c_2$ is strictly monotone between the extremals a and b only if f is (and hence $\{1, f\}$ forms a Chebyshev system). However, we did construct an example $\{1, h_2, f\}$ which we now present.

EXAMPLE. Let $h_2(x) = x$, $f(x) = x^3$ and $[a, b] = [-\frac{1}{2}, 1]$. Then $\{1, x, x^3\}$ is not a Chebyshev system on $[-\frac{1}{2}, 1]$ since, for example, $p(x) = x(x^2 - \frac{1}{16})$ has zeros at $-\frac{1}{4}$, 0, $\frac{1}{4}$. We will now show S_t exists such that properties (a)–(g) are satisfied. Letting $-\frac{1}{2} \leqslant t < 1$, $E_t(x) = x^3 - (a_t + b_t x)$ and using t, x_t, 1 as a reference set gives the equations

$$\begin{aligned} E_t(t) &= t^3 - (a_t + b_t t) = d_t, \\ E_t(x_t) &= x_t^3 - (a_t + b_t x_t) = -d_t, \\ E_t(1) &= 1 - (a_t + b_t) = d_t. \end{aligned} \tag{1}$$

Subtracting the third equation from the first equation gives $t^3 - 1 - b_t(t-1) = 0$, i.e., $b_t = t^2 + t + 1$. Now

$$\frac{d}{dt} E_t(x) = 3x^2 - b_t = 3x^2 - (t^2 + t + 1) = 0, \qquad \text{when} \quad x = x_t.$$

Hence, $x_t = [(t^2 + t + 1)/3]^{1/2}$. Substituting x_t and b_t into Eqs. (1), one could solve uniquely for a_t and d_t in terms of t and observe that $d_t > 0$; we omit the details. Considering dE_t/dx and using $t \geqslant -\frac{1}{2}$ we see $E_t(x)$ is strictly decreasing in $[t, x_t]$ and strictly increasing in $[x_t, 1]$. Hence, the characterization theorem guarantees that $a_t + b_t x$ obtained from solving (1) is the unique best uniform approximation to x^3 on $[t, 1]$.

Then, for $-\frac{1}{2} \leqslant t < 1$, $S_t(x) = (1/d_t)[x^3 - (a_t + b_t x)]$ satisfies (a)–(f). Now, let $-\frac{1}{2} \leqslant t_1 < t_2 < 1$. Since x_t is strictly increasing as a function of t, $x_{t_1} < x_{t_2}$. Clearly $S_{t_1}(x) - S_{t_2}(x)$ has a zero in (x_{t_1}, x_{t_2}) and a zero at $x = 1$. If $S_{t_1} - S_{t_2}$ has no other zeros in $[t_2, 1]$, then (g) will be satisfied. Assume the opposite; then by Rolle's theorem $d[S_{t_1}(x) - S_{t_2}(x)]/dx$ has at least two zeros, say $z_1 < z_2$, in $(t_2, 1)$ with $z_2 > x_{t_1} \geqslant x_{-1/2} = \frac{1}{2}$. Hence, $z_2 \neq -z_1$ which is impossible since $d[S_{t_1}(x) - S_{t_2}(x)]/dx$ has the form $c_1 x^2 + c_2$. This completes the verification of (a)–(g) for the example.

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